LOCALLY UNIFORM NO-COLLISION SUSPENSIONS IN EULER APPROXIMATION

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Pseudoturbulent motions of phases are considered as well as the internal structure of monodispersional locally uniform suspension of particles in a liquid when the derivatives of the dynamic variables which describe the mean flow of the suspension are ignored. One assumes that the number of collision dissipations, which expresses the ratio of the dissipation force due to acceleration of the fluid phase in the case of stepwise change in the velocities of the colliding particles to the forces of viscous interaction, is small. Dynamic equations which determine the motion of the suspension in its continuous approximation and the balance equation of the pseudoturbulent energy of the particles are obtained with an approximation which is similar to the Euler approximation in the hydrodynamics of a single-phase medium.

1. Spectral Measures of Pseudoturbulent Random Processes. In [1] the stochastic equations were obtained for the pseudoturbulent pulsations of the volume concentration of the suspension ρ', the pressure p', velocities of the fluid $\mathbf{v}'$ and the particle $\mathbf{w}'$, and the resulting algebraic equations for spectral measures of the random functions. It is our aim to obtain a complete system of equations which determine the mean motion of the suspension in the "Euler approximation," that is, with an accuracy up to the terms of zero order as regards the ratio of the scales of pseudoturbulence to the corresponding scales of the mean motion. To this end, the equations for the spectral measures are written neglecting the derivatives of the dynamic variables:

$$(\omega + \mathbf{uk})\,dZ_\rho - (1-\rho)\,\mathbf{k}d\mathbf{Z}_v = 0$$

$$id_1(1-\rho)(\omega + \mathbf{uk})\,d\mathbf{Z}_v = -i(1-\rho)\,\mathbf{k}dZ_p - \mu_0 S\left(k^2 d\mathbf{Z}_v + {}^1/_3\,\mathbf{k}\,(\mathbf{k}d\mathbf{Z}_v)\right)$$
$$- d_1\rho\,[\beta K + i\xi\omega + \gamma\eta\,(1 + i\,\mathrm{sign}\,\omega)\sqrt{|\omega|}\,]\,(d\mathbf{Z}_v - d\mathbf{Z}_w) - d_1\rho\beta K'\mathbf{u}dZ_\rho$$

$$id_2\omega d\mathbf{Z}_w = -i\mathbf{k}dZ_p + d_1\,[\beta K + i\xi\omega + \gamma\eta\,(1 + i\,\mathrm{sign}\,\omega)\sqrt{|\omega|}\,]\,(d\mathbf{Z}_v - d\mathbf{Z}_w)$$
$$+ d_1\beta K'\mathbf{u}dZ_\rho - d_1\xi d\mathbf{Z}_w, \quad K' \equiv dK/d\rho \tag{1.1}$$

$$\beta = \frac{9\nu_0}{a^2}, \quad \gamma = \frac{9}{2a}\left(\frac{\nu_0}{\pi}\right)^{1/2}, \quad \xi = \frac{c}{4}\frac{N}{n}\left[\xi + 4\gamma\eta\left(\frac{n}{N}\right)^{1/2}\right]$$

The notation used in [1, 2] was adopted, though the symbol ⟨ ⟩ in the notation for dynamic variables was omitted. In (1.1) a representation was employed for the interphase interaction force which is valid for $R \ll 1$, where $R = 2au/\nu_0$ is the Reynolds number. Moreover, it is assumed below that the flow is locally uniform in the sense that in it large particle aggregates are not formed, or cavities filled by the dispersion medium, etc.

Dimensionless variables are now introduced as in [2] by using the formulas

$$d\mathbf{Z}_v' = \frac{d\mathbf{Z}_v}{u}, \quad d\mathbf{Z}_w' = \frac{d\mathbf{Z}_w}{u}, \quad dZ_p' = \frac{dZ_p}{d_1\beta K u a}, \quad \omega' = \frac{a\omega}{u}, \quad \mathbf{k}' = a\mathbf{k} \tag{1.2}$$

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The following dimensionless equations are obtained from (1.1) and (1.2) after transformations:

$$(\omega' + \mathbf{u}_0\mathbf{k}')\, dZ_\rho - (1-\rho)\,\mathbf{k}' d\mathbf{Z}_v' = 0 \tag{1.3}$$

$$i\frac{r}{\varkappa}\omega' d\mathbf{Z}_w' = -i\mathbf{k}' dZ_p' + \left[1 + ir\xi\omega' + \frac{3}{\sqrt{2\pi}}\eta(1 + i\,\mathrm{sign}\,\omega)\sqrt{r|\omega'|}\right]$$

$$\times (d\mathbf{Z}_v' - d\mathbf{Z}_w') + \frac{d\ln K}{d\rho}\mathbf{u}_0 dZ_\rho - \alpha d\mathbf{Z}_w'$$

$$\rho\left(i\frac{r}{\varkappa}\omega' + \alpha\right) d\mathbf{Z}_w' + [ir(1-\rho)(\omega' + \mathbf{u}_0\mathbf{k}') + sk'^2]\, d\mathbf{Z}_v' = -i\mathbf{k}' dZ_p' - {}^1/_3\, s\mathbf{k}'(\mathbf{k}' d\mathbf{Z}_v')$$

$$\varkappa = \frac{d_1}{d_2}, \quad \mathbf{u}_0 = \frac{\mathbf{u}}{u}, \quad s = \frac{2}{9}\frac{S}{K}, \quad r = \frac{R}{9K} = \frac{2}{9K}\frac{ua}{\nu_0} \ll 1, \quad \alpha = \frac{\xi}{\beta K}$$

For $\omega' = 0$ Eqs. (1.3) reduce to those given in [2].

The parameter α in (1.3) characterizes the ratio of dissipative forces due to acceleration of the additional fluid masses in the case of stepwise changes in the particle velocities at the time of the collisions to the force of viscous interaction between the phases of the suspension [1]. If the concentration ρ of the suspension is not too close to the concentration ρ_* of the layer of particles in a densely packed state, one can consider the "number of collision dissipations" α as small (see the discussion in [2]). Below the suspension is considered only when direct particle collisions are disregarded, that is, for $\alpha = 0$.

Disregarding the parameters α and r in (1.3), we obtain the solution of these equations in the form

$$-i\mathbf{k}' dZ_p' = \frac{dZ_\rho}{i(r/\varkappa)(1-\rho)\omega' + 1}\left[i\frac{r}{\varkappa}\omega'\left(\rho\frac{d\ln K}{d\rho}\mathbf{u}_0\mathbf{k}' + \left(\rho + \frac{4}{3}sk'^2\right)\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right) + \frac{4}{3}sk'^2\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right]\frac{\mathbf{k}'}{k'^2} \tag{1.4}$$

$$d\mathbf{Z}_v' = \left[i\frac{r}{\varkappa}\omega'(\rho + sk'^2) + sk'^2\right]^{-1}\left[-\left(i\frac{r}{\varkappa}(1-\rho)\omega' + 1\right)i\mathbf{k}' dZ_p' - \left(i\frac{r}{\varkappa}\omega'\left(\rho\frac{d\ln K}{d\rho}\mathbf{u}_0 + \frac{s}{3}\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\mathbf{k}'\right) + \frac{s}{3}\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\mathbf{k}'\right) dZ_\rho\right]$$

$$d\mathbf{Z}_w' = \left[i\frac{r}{\varkappa}\omega'(\rho + sk'^2) + sk'^2\right]^{-1}\left[-(1 + sk'^2)\,i\mathbf{k}' dZ_p' + sk'^2\left(\frac{d\ln K}{d\rho}\mathbf{u}_0 - \frac{1}{3}\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}\right) dZ_\rho\right]$$

In the general case, the computations which are based on the relations (1.4) are very complex and cumbersome. One can easily see, however, that two much simpler cases are feasible: 1) $\varkappa \gtrsim 1$, when $r/\varkappa \ll 1$, and 2) $\varkappa \lll 1$, such that $r/\varkappa \gg 1$. To be specific, suspensions or emulsions will be considered below only in dripping fluids when $r/\varkappa \ll 1$. An independent analysis is required for the other case, which is characteristic for suspensions of matter in gas with r not too small.

From (1.4) one obtains for $r/\varkappa \ll 1$ the relations

$$-i\mathbf{k}' dZ_p' = \frac{4}{3}sk'^2\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}dZ_\rho, \quad d\mathbf{Z}_v' = \frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}dZ_\rho \tag{1.5}$$

$$d\mathbf{Z}_w' = \left[\frac{d\ln K}{d\rho}\mathbf{u}_0 + \left(1 + \frac{4}{3}sk'^2\right)\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}\right] dZ_\rho$$

The corresponding expressions for dimensionless spectral densities are of the form

$$\mathbf{\Psi}'_{\rho,v}(\omega', \mathbf{k}') = \frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\mathbf{\Psi}'_{\rho,w}(\omega', \mathbf{k}') = \left[\frac{d\ln K}{d\rho}\mathbf{u}_0 + \left(1 + \frac{4}{3}sk'^2\right)\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}\right]\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\mathbf{\Psi}'_{v,v}(\omega', \mathbf{k}') = \left(\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right)^2\frac{\mathbf{k}' * \mathbf{k}'}{k'^4}\Psi'_{\rho,\rho}(\omega', \mathbf{k}'), \quad \mathbf{a} * \mathbf{b} = \|a_i b_j\| \tag{1.6}$$

$$\mathbf{\Psi}'_{w,w}(\omega', \mathbf{k}') = \left[\left(\frac{d\ln K}{d\rho}\right)^2\mathbf{u}_0 * \mathbf{u}_0 + \left(1 + \frac{4}{3}sk'^2\right)\frac{d\ln K}{d\rho}\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right.$$

$$\left.\times\frac{\mathbf{u}_0 * \mathbf{k}' + \mathbf{k}' * \mathbf{u}_0}{k'^2} + \left(1 + \frac{4}{3}sk'^2\right)\left(\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right)^2\frac{\mathbf{k}' * \mathbf{k}'}{k'^4}\right]\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\mathbf{\Psi}'_{v,w}(\omega', \mathbf{k}') = \left[\frac{d\ln K}{d\rho}\mathbf{k}' * \mathbf{u}_0 + \left(1 + \frac{4}{3}sk'^2\right)\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}' * \mathbf{k}'}{k'^2}\right]\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{(1-\rho)k'^2}\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\mathbf{\Psi}'_{-\nabla p,\rho}(\omega', \mathbf{k}') = \frac{4}{3}sk'^2\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\frac{\mathbf{k}'}{k'^2}\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\Psi'_{-\nabla p,v}(\omega', \mathbf{k}') = \frac{4}{3} sk'^2 \left(\frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho}\right)^2 \frac{\mathbf{k}' * \mathbf{k}'}{k'^4} \Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

$$\Psi'_{-\nabla p,w}(\omega', \mathbf{k}') = \frac{4}{3} sk'^2 \left[\frac{d\ln K}{d\rho} \mathbf{k}' * \mathbf{u}_0 + \left(1 + \frac{4}{3} sk'^2\right) \frac{\omega' + \mathbf{u}_0\mathbf{k}'}{1-\rho} \frac{\mathbf{k}' * \mathbf{k}'}{k'^2}\right] \frac{\omega' + \mathbf{u}_0\mathbf{k}'}{(1-\rho) k'^2} \Psi'_{\rho,\rho}(\omega', \mathbf{k}')$$

The quantity $\Psi'_{\rho,\rho}(\omega', \mathbf{k}')$ can be represented as [1, 2]

$$\Psi'_{\rho,\rho}(\omega', \mathbf{k}') = \frac{\mathbf{D}'\mathbf{k}'\mathbf{k}'}{\pi} \frac{\Phi'_{\rho,\rho}(\mathbf{k}')}{M(\omega', \mathbf{k}')}, \quad \Phi'_{\rho,\rho}(\omega', \mathbf{k}') = \frac{\Phi}{k_0'^3} Y(k_0' - k')$$

$$M(\omega', \mathbf{k}') = \omega'^2 + \left(\mathbf{D}'\mathbf{k}'\mathbf{k}' - \frac{\operatorname{tr}\mathbf{D}}{\theta_0}\omega'^2\right), \quad \mathbf{D}' = \frac{\mathbf{D}}{ua}, \quad \theta_0 = \frac{\langle w'^2\rangle}{u^2} \tag{1.7}$$

$$\Phi = \frac{3}{4\pi}\rho^2\left(1 - \frac{\rho}{\rho^*}\right), \quad k_0' = \left(\frac{3\pi\rho}{2}\right)^{1/3}\left(1 - \frac{\rho}{\rho^*}\right)^{-1/3}, \quad Y(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

It should be mentioned that the expression for k_0' in (1.7) is only valid for the locally uniform suspensions under consideration. In (1.7) $\mathbf{D}$ represents the tensor of pseudoturbulent particle diffusion which was considered in detail in [2]; $\langle w'^2\rangle$ is the mean square of their pulsation velocity. The relations (1.6) and (1.7) enable one, of course, to compute various correlation functions of pseudoturbulence in the "equilibrium" state. In contrast to the kinetic gas theory in which by equilibrium one understands the spatially uniform stationary state, here by equilibrium state one understands, in general, a state in which the particle-distribution function f of velocities of the pseudoturbulent pulsations has a "quasi-Maxwellian" form (see Sec. 4). This concept is explained in detail below. It is noted, however, that both these definitions of an equilibrium are identical within the framework of the Euler approximation which is considered here.

An equilibrium state of a suspension can also be formally defined as a state in which the relations (1.7) are valid for the spectral density of the random process ρ' and also from the expression $\langle\rho'^2\rangle = \rho^2(1-\rho/\rho_*)$, which follows from it.

2. Equilibrium Structure of Suspensions. In our considerations only the averages of binary products of different pseudoturbulent quantities are required. It is convenient to direct the coordinate axis x_1 in the direction of the unit vector $\mathbf{u}_0$ which specifies the symmetry axis of the pseudoturbulent motions.

1°. Intensity of particle pulsations. Integrating $\Psi'_{ww}(\omega', \mathbf{k}')$ over all frequencies ω' and the entire wave space $\mathbf{k}'$ and proceeding to dimensionsal variables one obtains the following expressions for the nonvanishing averages $\langle w_i' w_j'\rangle$:

$$\langle w_1'^2\rangle = \frac{4\pi\Phi}{3(1-\rho)^2}\left[n^2(\rho) + \frac{2}{3}n(\rho)\left(1 + \frac{4}{5}sk_0'^2\right) + \frac{1}{5}\left(1 + \frac{8}{5}sk_0'^2 + \frac{16}{21}s^2k_0'^4\right)\left(1 + \frac{1 + 5/3\,\gamma^2}{1 + 3\gamma^2}\theta_0\right)\right]u^2, \tag{2.1}$$

$$\gamma^2 = \frac{D_2}{D_1 - D_2} > 0, \quad n(\rho) = (1-\rho)\frac{d\ln K}{d\rho}$$

$$\langle w_2'^2\rangle \equiv \langle w_3'^2\rangle = \frac{4\pi\Phi}{45(1-\rho)^2}\left(1 + \frac{8}{5}sk_0'^2 + \frac{16}{21}s^2k_0'^4\right)\left(1 + \frac{1 + 5\gamma^2}{1 + 3\gamma^2}\theta_0\right)u^2$$

The quantity γ^2 was considered in [2]; usually, $\gamma^2 \approx 10^{-2}$-10^{-3}; therefore for simplicity it is assumed that $\gamma \approx 0$. Adding up the expressions (2.1) one obtains an equation for the unknown θ_0. By solving it one obtains for θ_0 the expression

$$\theta_0 = \frac{4\pi\Phi}{3(1-\rho)^2}\left[n^2(\rho) + \frac{2}{3}n(\rho)\left(1 + \frac{4}{5}sk_0'^2\right) + \frac{1}{3}\left(1 + \frac{8}{5}sk_0'^2 + \frac{16}{21}s^2k_0'^4\right)\right]\left[1 - \frac{4\pi\Phi}{9(1-\rho)^2}\left(1 + \frac{8}{5}sk_0'^2 + \frac{16}{21}s^2k_0'^4\right)\right]^{-1} \tag{2.2}$$

It is not difficult to see that θ_0 increases without bounds for $\rho \to \rho°$, where $\rho°$ is the root of the equation

$$\frac{4\pi\Phi}{9(1-\rho)^2}\left(1 + \frac{8}{5}rk_0'^2 + \frac{16}{21}s^2k_0'^4\right) = 1 \tag{2.3}$$

The obtained divergence results from the fact that collision dissipations have been disregarded by us; this ceases to be valid in the immediate proximity to the densely packed state. It is obvious, therefore, that the theory developed here can anyway be valid for $\rho < \rho°$; however, it will be seen that $\rho°$ differs very slightly from ρ_*.

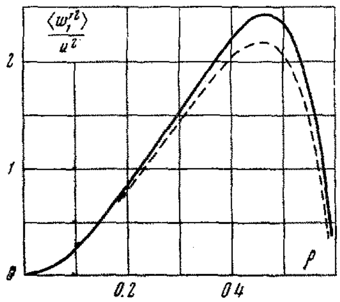

Fig. 1

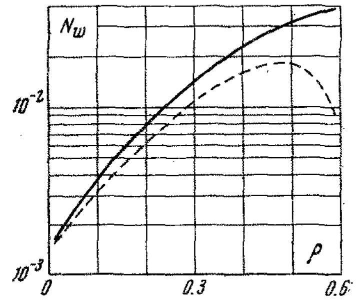

Fig. 2

The following approximation was used for numerical calculations in [2]:

$$K(\rho) \approx \begin{cases} (1-\rho)^{-4.58}, & \rho < \rho_0 = 0.28 \\ 25\rho\,(3(1-\rho)^2)^{-1}, & \rho > \rho_0 \end{cases}, \quad S(\rho) \approx (1-\rho)^{-2.5} \tag{2.4}$$

This approximation of $K(\rho)$ is somewhat inconvenient in that the second derivative of $K(\rho)$ has a discontinuity at the point $\rho = \rho_0$; therefore, for functions of ρ there appear corner points for $\rho = \rho_0$ and in some cases even discontinuities. Therefore by using the same expression (2.4) for $S(\rho)$ we shall use $K(\rho)$ below in the form

$$K(\rho) = \frac{2.2}{(1-\rho)^{2.9}} - 1.2, \quad n(\rho) = \frac{6.38}{2.2 - 1.2\,(1-\rho)^{2.9}} \tag{2.5}$$

A comparison was made between the values of $K(\rho)$ in (2.4) and in (2.5) in the ρ interval from 0 to $\rho_* = 0.60$:

$\rho = 0.10$	0.20	0.30	0.40	0.50	0.60
$K(2.4) = 1.6202$	2.7787	5.1020	9.2593	16.6667	31.2500
$K(2.5) = 1.7862$	3.0021	4.9893	8.4780	15.2214	30.1652

It can be seen that the function (2.5) provides a good approximation of $K(\rho)$ of (2.4) though it possesses continuous derivatives.

The calculation of the root ρ° of Eq. (2.3) for $\rho_* = 0.60$ yields the inequalities $0.599997 < \rho^\circ < 0.599998$, that is, ρ° is almost identical with ρ_*.

In Figs. 1 and 2 the expressions $\langle w_1'^2\rangle / u^2$ and $N_W = \langle w_2'^2\rangle / \langle w_1'^2\rangle$ as functions of ρ are given as continuous curves. The dashed curves correspond to the functions resulting from an "inviscous" model $s = 0$ (see [2]). Above in the calculations of $S(\rho)$ (2.4) and $K(\rho)$ (2.5), we have used $\gamma \approx 0$, $\rho_* = 0.60$, which we will also use below.

It is noted that the obtained results refer only to suspensions of solid particles. For trickles or little cavities, different representations must obviously be used for $S(\rho)$ and $K(\rho)$.

2°. Other pseudoturbulent averages. The following expressions are obtained from (1.6) and (1.7):

$$\langle v_1'^2\rangle = \frac{4\pi\Phi}{15\,(1-\rho)^2}\left(1 + \frac{1 + {}^5/_3\gamma^2}{1+3\gamma^2}\theta_0\right)u^2, \quad \langle v_2'^2\rangle \equiv \langle v_3'^2\rangle = \frac{4\pi\Phi}{45\,(1-\rho)^2} \times \left(1 + \frac{1+5\gamma^2}{1+3\gamma^2}\theta_0\right)u^2$$

$$\langle v_1'w_1'\rangle = \frac{4\pi\Phi}{9\,(1-\rho)^2}\left[n(\rho) + \frac{3}{5}\left(1 + \frac{4}{5}sk_0'^2\right)\left(1 + \frac{1+{}^5/_3\gamma^2}{1+3\gamma^2}\theta_0\right)\right]u^2$$

$$\langle v_2'w_2'\rangle \equiv \langle v_3'w_3'\rangle = \frac{4\pi\Phi}{45\,(1-\rho)^2}\left(1 + \frac{4}{5}sk_0'^2\right)\left(1 + \frac{1+5\gamma^2}{1+3\gamma^2}\theta_0\right)u^2$$

$$\langle \rho'v_i'\rangle = \frac{4\pi\Phi u}{9\,(1-\rho)}\delta_{i1}, \quad \langle \rho'w_i'\rangle = \frac{4\pi\Phi}{3\,(1-\rho)}\left[n(\rho) + \frac{1}{3}\left(1 + \frac{4}{5}sk_0'^2\right)\right]u\delta_{i1}$$

$$\left\langle -\frac{\partial p'}{\partial x_i}\rho'\right\rangle = \frac{16\pi\Phi sk_0'^2}{45\,(1-\rho)}d_1\beta K u\delta_{i1} \tag{2.6}$$

$$\left\langle -\frac{\partial p'}{\partial x_1}v_1'\right\rangle = \frac{16\pi\Phi sk_0'^2}{75\,(1-\rho)^2}\left(1 + \frac{1+{}^5/_3\gamma^2}{1+3\gamma^2}\theta_0\right)d_1\beta K u^2$$

$$\left\langle -\frac{\partial p'}{\partial x_2}v_2'\right\rangle \equiv \left\langle -\frac{\partial p'}{\partial x_3}v_3'\right\rangle = \frac{16\pi\Phi sk_0'^2}{225\,(1-\rho)^2}\left(1 + \frac{1+5\gamma^2}{1+3\gamma^2}\theta_0\right)d_1\beta K u^2$$

$$\left\langle -\frac{\partial p'}{\partial x_1}w_1'\right\rangle = \frac{16\pi\Phi sk_0'^2}{45\,(1-\rho)^2}\left[n(\rho) + \left(\frac{3}{5} + \frac{4}{7}sk_0'^2\right)\left(1 + \frac{1+{}^5/_3\gamma^2}{1+3\gamma^2}\theta_0\right)\right]d_1\beta K u^2$$

$$\left\langle -\frac{\partial p'}{\partial x_2} w_2' \right\rangle = \left\langle -\frac{\partial p'}{\partial x_3} w_3' \right\rangle = \frac{16\pi\Phi s k_0'^2}{225(1-\rho)^2}\left(1+\frac{20}{21} s k_0'^2\right)\left(1+\frac{1+5\gamma^2}{1+3\gamma^2}\theta_0\right) d_1 \beta K u^2$$

$$\langle v_i' v_j' \rangle = \langle w_i' w_j' \rangle = \langle v_i' w_j' \rangle = \left\langle -\frac{\partial p'}{\partial x_i} v_j' \right\rangle = \left\langle -\frac{\partial p'}{\partial x_i} w_j' \right\rangle = 0, \quad i \neq j$$

A direct use of the relations (1.5) yields also

$$\left\langle \rho' \frac{dv_i'}{dt} \right\rangle = \left\langle \rho' \frac{du_i'}{dt} \right\rangle = \left\langle v_i' \frac{dv_j'}{dt} \right\rangle = \left\langle w_i' \frac{dv_j'}{dt} \right\rangle = \left\langle v_i' \frac{dw_j'}{dt} \right\rangle = \left\langle w_i' \frac{dw_j'}{dt} \right\rangle = 0 \tag{2.7}$$

Differentiation with respect to time is carried out along a particle trajectory [1].

It is not difficult to see that for $\gamma \approx 0$ the quantity $N_v = \langle v_2'^2 \rangle / \langle v_1'^2 \rangle \approx 1/3$; and the functions $\langle v_1'^2 \rangle / u^2$ and $\langle \rho' v_1' \rangle / u$ of ρ are as shown in Fig. 3 (curves 1 and 2, respectively). The dashed line illustrates the first of these quantities for $s = 0$. Other pseudoturbulent averages as functions of ρ are of approximately the same character and are therefore not shown.

3°. Average force of the interphase interaction. The average interaction force between phases was computed in [1]. Disregarding, for simplicity, the possible dependence of the coefficients ξ and η on the concentration ρ, we write down this force per single particle as

$$\mathbf{F}_i = -\sigma_0 \frac{\partial p}{\partial \mathbf{r}} + \varkappa m \beta K^* \mathbf{u} + \varkappa m \left(\xi \frac{D\mathbf{u}}{Dt} + \gamma \int_{-\infty}^{t} \eta \frac{D\mathbf{u}}{Dt}\bigg|_{t=t'} \frac{dt'}{\sqrt{t-t'}} \right) \tag{2.8}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{w} \frac{\partial}{\partial \mathbf{r}},$$

$$K^* = K + \frac{\langle \rho' u' \rangle}{u} \frac{dK}{d\rho} + \frac{1}{2} \langle \rho'^2 \rangle \frac{d^2 K}{d\rho^2}, \quad \mathbf{u}' = \mathbf{v}' - \mathbf{w}'$$

The second term of (2.8) represents the force of viscous interphase interaction, whose magnitude differs considerably in a system of chaotically moving particles from the magnitude characteristic in a system with relatively stationary particles. From (2.6) and (2.8) the relation is obtained in the equilibrium state:

$$K^* = \lambda_K K, \quad \lambda_K = 1 + \frac{4\pi\Phi}{3(1-\rho)^2} \left[\frac{(1-\rho)^2}{2K} \frac{d^2 K}{d\rho^2} - n(\rho) \left(n(\rho) + \frac{4}{15} s k_0'^2 \right) \right] \tag{2.9}$$

The above introduced velocity of interphase slip u is not identical with the effective velocity u^* in the intervals between the particles determined by the full relative flow Q of the fluid phase. Indeed,

$$Q = (1-\rho) u^* = (1-\rho) u - \langle \rho' v_1' \rangle, \quad u^* = \lambda_u u, \quad \lambda_u = 1 - \frac{4\pi\Phi}{9(1-\rho)^2} \tag{2.10}$$

In experiments the force of viscous interaction is usually expressed either by Q or in terms of u^*, for example, $F_1 = \varkappa m \beta K_1(\rho) u^*$. One then obtains from (2.9) and (2.10)

$$K^* = (\lambda_K / \lambda_u) K_1 \tag{2.11}$$

In Eq. (2.11) the coefficient indicates what portion of the layer hydraulic resistance is taken by an equivalent in porosity layer of particles which perform a full pseudoturbulent motion.

In Fig. 4 the quantity λ_K is shown as a function of ρ for $s \neq 0$ or $s = 0$ (dashed line). The corresponding curves for λ_K/λ_u are only slightly above the curves for λ_K. It is not difficult to see that λ_K and λ_K/λ_u are always less than unity.

The latter enables one to explain quite simply the phenomenon known as "the effect of lower resistance of a pseudoliquid layer" (see, for example, [3-6]). Endeavors have repeatedly been made to explain this phenomenon, which is important in practice and which is usually connected with the observed weak circulation of particles in the layer [7, 8]. The role of fluctuations of porosity $\varepsilon = 1 - \rho$ of the layer in the lowering of its hydraulic resistance was apparently mentioned for the first time in [6, 9]; an explanation similar to the one given above can also be found in [10].

It is noted that the relations (2.9)-(2.11) refer to a system in equilibrium state with pseudoturbulence fully developed. In actual systems a considerable effect of the flow boundaries and in particular of the internal circulation of phases on the effective hydraulic resistance must obviously be expected.

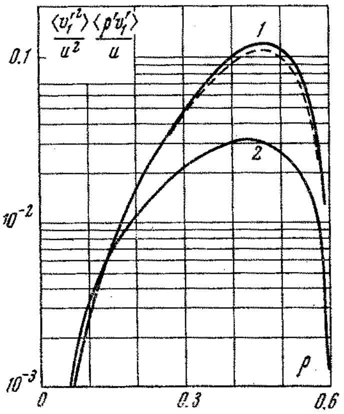

Fig. 3

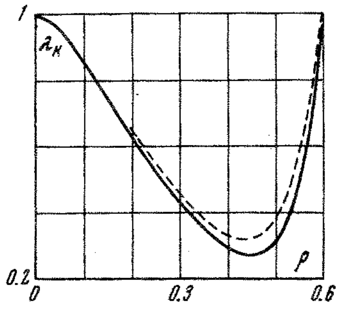

Fig. 4

The analysis of the results obtained in 1°-3° shows that good results are also obtained for suspensions of moderate concentration if the simplified model $s = 0$ is used. This model differs considerably from that of $s \neq 0$ only for very large ρ. It is also noted that the anisotropy degree of pseudoturbulent motions rapidly diminishes with increasing ρ.

In fact, an increase in the pseudoturbulent averages when $\rho \to \rho°$ takes place only in the region of $\rho > 0.58$ and is not shown in Figs. 1-3.

3. Non-equilibrium Structure of Suspensions and Dynamic Equations. The relations obtained in Sec. 2 characterize pseudoturbulence of suspensions away from the surfaces bounding its motion, for example, of the walls, grids, free surfaces, etc., under the assumption that expressions (1.7) are valid. In fact, the boundaries of flow as well as its nonstationarity or lack of uniformity can obviously have a considerable effect on pseudoturbulence intensity and infringe, in particular, (1.7). For example, rigid walls contribute to the damping of pseudoturbulent pulsations; grids let through liquid phase but not solid particles. Depending on the degree of nonuniformity in the fluid flow the grids can either weaken or strengthen the pseudoturbulence near a grid. This effect may in a number of cases extend to a considerable distance from the boundary.

An attempt was made in [10] to allow for this "nonuniformity" by assuming that it has no adverse effect on the relation (1.7), that is, on the magnitude of fluctuations in suspension concentration. In this case the equations for various pseudoturbulent averages and in particular for pulsation energies of phases in different directions are essentially obtained in the same manner as the equations for correlation functions in statistical mechanics of turbulence. Also different averages may depend on the coordinates and on time in quite different manner.

Here another, much simpler model will be considered whose formulation does not need any additional assumptions; namely, one takes into account that in [1] Eqs. (1.1) were obtained after averaging over $\Delta t \gg \tau$, where τ is a characteristic least time pseudoturbulence scale which is identical with the time of internal interactions in the system and results in establishing local equilibrium state; the latter is similar in a sense to the state of molecular chaos in nonuniform and nonstationary gas flow (see also the discussion in [10]). One can therefore employ Eq. (1.1) not only in the analysis of equilibrium pseudoturbulence but also of the local equilibrium pseudoturbulence; only the latter will be considered below.

As before, Eqs. (1.1) permit to express all spectral measures in terms of a single one; these expressions have the same form in a nonequilibrium state as in an equilibrium state. Therefore if one has a nonequilibrium state $\langle w'^2\rangle = \theta$, then for any pseudoturbulent variables in this state one has

$$\langle a'b'\rangle = (\theta / \theta_0)\langle a'b'\rangle_0 \tag{3.1}$$

Above and also everywhere below, the subscript zero refers to the quantities which correspond to the equilibrium state and which were computed in Sec. 2. In particular, it follows clearly from (3.1) that the ratios such as $\langle a'b'\rangle / \langle c'b'\rangle$ are the same in an equilibrium as well as in a corresponding nonequilibrium state. Much advantage is taken of this fact in our further considerations.

Thus the problem of describing a nonequilibrium (but locally in equilibrium) pseudoturbulence reduces in fact to an additional equation for single scalar quantity θ. This equation is considered in Sec. 4.

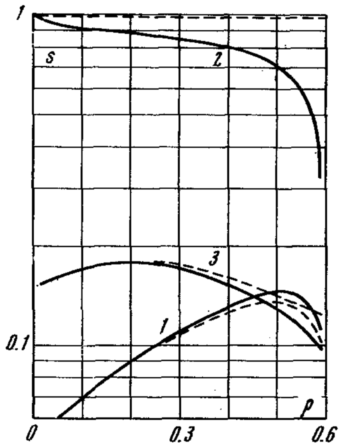

Fig. 5

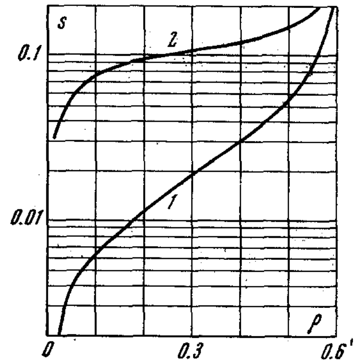

Fig. 6

Using the results of [1], one can represent dynamic equations in the form

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial \mathbf{r}}(\rho \mathbf{w})=0, \quad \frac{\partial \rho}{\partial t}-\frac{\partial}{\partial \mathbf{r}}((1-\rho)\mathbf{v}+\mathbf{q})=0, \quad \mathbf{q}=-\langle\rho'\mathbf{v}'\rangle$$

$$d_2\rho\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}=-\frac{\partial \mathbf{P}^{(p)}}{\partial \mathbf{r}}+\frac{\rho}{\sigma_0}\mathbf{F}_i+d_2\rho\mathbf{q}, \quad \mathbf{P}^{(p)}=d_2\rho\langle\mathbf{w}' * \mathbf{w}'\rangle$$

$$d_1\left[\frac{\partial}{\partial t}((1-\rho)\mathbf{v})+\frac{\partial}{\partial \mathbf{r}}((1-\rho)\mathbf{v}*\mathbf{v})+\frac{\partial \mathbf{q}}{\partial t}\right] \tag{3.2}$$

$$=-\frac{\partial \mathbf{P}^{(f)}}{\partial \mathbf{r}}-\frac{\partial p}{\partial \mathbf{r}}+\mu_0\frac{\partial}{\partial \mathbf{r}}\left(S\mathbf{e}+\frac{1}{2}\langle\rho'^2\rangle\frac{d^2S}{d\rho^2}\mathbf{e}\right)-\frac{\rho}{\sigma_0}\mathbf{F}_i+d_1(1-\rho)\mathbf{q}$$

$$\mathbf{e}=\left\|\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial x_l}\right\|, \quad \mathbf{P}^{(f)}=d_1\left[(1-\rho)\langle\mathbf{v}' * \mathbf{v}'\rangle+\mathbf{q}*\mathbf{v}+\mathbf{v}*\mathbf{q}\right]$$

When writing out (3.2), the identity $\langle\rho'\mathbf{e}'\rangle\equiv 0$ has been taken into account; it follows directly from (1.5). Ignoring the fact that the coefficients ξ and η depend on ρ one obtains the relations (2.8) for $\mathbf{F}_i$. It is emphasized again that the pseudoturbulent quantities appearing in (3.2) are not, in general, identical with their equilibrium values determined previously but should be computed according to (3.1).

4. Equilibrium Distribution Function and Transfer Equation of Pseudoturbulent Energy of Particles. For a system of suspended particles a kinetic equation was formulated in [1]. In the case under consideration it can be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)f+\mathbf{w}'\frac{\partial f}{\partial \mathbf{r}}+\frac{\partial}{\partial \mathbf{w}'}\left[\left(\mathbf{q}+\frac{\mathbf{F}_i+\mathbf{F}_i''}{m}-\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}\right)f\right]-\left(\frac{\partial f}{\partial \mathbf{w}'}*\mathbf{w}'\right):\left(\frac{\partial}{\partial \mathbf{r}}*\mathbf{w}\right)$$

$$=\frac{\mathbf{A}}{m}:\left(\frac{\partial}{\partial \mathbf{w}'}*\frac{\partial}{\partial \mathbf{w}'}\right)f, \quad \mathbf{A}:\mathbf{B}=A_{ij}B_{ji} \tag{4.1}$$

In the above $\mathbf{F}_i''$ are fluctuations in the interaction forces between the particles and the fluid phase which are averaged in accordance with conditional distributions using the method in [1]; $\mathbf{A}$ is an unknown tensor which describes the diffusion in the velocity space. The collision term on the right-hand side of (4.1) has been omitted in view of the fact that direct collisions of particles have been disregarded by us.

Employing the relations (2.6) and (2.7) as well as the general method in [1], we obtain the following expressions after calculations:

$$\mathbf{F}_i''=\varkappa m\beta\left[K(s_{u,ii}+s'_{-\nabla p,ii})\mathbf{w}'+\frac{dK}{d\rho}s'_{\rho,1}w_1\mathbf{u}_0\right]$$

$$s_{u,ii}=s_{v,ii}-1, \quad s_{v,ii}=\frac{\langle v_i'w_i'\rangle}{\langle w_i'^2\rangle}, \quad s'_{\rho,1}=us_{\rho,1}=u\frac{\langle\rho'w_1'\rangle}{\langle w_1'^2\rangle}, \tag{4.2}$$

$$s'_{-\nabla p,ii}=\frac{s_{-\nabla p,ii}}{d_1\beta K\langle w_i'^2\rangle}=\frac{1}{d_1\beta K\langle w_i'^2\rangle}\left\langle-\frac{\partial p'}{\partial x_i}w_i'\right\rangle$$

Here no summation over i is carried out; for simplicity, one ignores the fact that ξ and η are functions of ρ.

It is not difficult to see that the ratios of the averages of products of the pseudoturbulent quantities in nonequilibrium and equilibrium states, introduced in (4.2), are identical (see also Sec. 3), that is, they are independent of θ and can be regarded as known functions of dynamic variables. The quantities $s_{v,11}$, $s_{v,22}$, and $s_{\rho,1}$ as functions of ρ, calculated in accordance with the results of Sec. 2 are shown in Fig. 5 (curves 1, 2, 3, respectively). The quantities $s'_{-\nabla p,11}$ and $s'_{-\nabla p,22}$ as functions of ρ are given by curves 1 and 2 of Fig. 6. The dashed curves in Fig. 5 correspond to the inviscid model $s=0$.

The tensor **A** in (4.1) can easily be determined by procedures described in [1]; namely, one considers the distribution function of particles in the equilibrium state $f^{(0)}$ when Eq. (4.1) together with (4.2) becomes

$$\frac{\partial}{\partial \mathbf{w}'}\left[\left(\mathbf{g}+\frac{\mathbf{F}_i}{m}+c_2\mathbf{w}'+(c_1-c_2)\,w_1'\mathbf{u}_0\right)f^{(0)}\right]=\frac{\mathbf{A}}{m}:\left(\frac{\partial}{\partial \mathbf{w}'}*\frac{\partial}{\partial \mathbf{w}'}\right)f^{(0)}$$
$$c_1=\varkappa\beta\left[K\left(s_{u,11}-s'_{-\nabla p,11}\right)+\frac{dK}{d\rho}s'_{\rho,1}\right] \tag{4.3}$$
$$c_2\equiv c_3=\varkappa\beta K\left(s_{u,22}-s'_{-\nabla p,22}\right)$$

In the above c_j are some known functions of the dynamic variables. However, in view of the axial symmetry of the pseudoturbulence it can be assumed at once that the tensor **A** is diagonal with the eigenvalues A_1, $A_2=A_3$; moreover, a solution of (4.3) is sought in the quasi-Maxwellian form

$$f^{(0)}=n\left(\frac{B_1B_2B_3}{\pi^3}\right)^{1/2}\exp\left(-\Sigma B_j w_j'^2\right),\qquad B_2\equiv B_3 \tag{4.4}$$

where n is a countable concentration of particles in the suspension.

By inserting (4.4) in (4.3) we arrive at the equations

$$\mathbf{g}+\frac{\mathbf{F}_i}{m}=0,\quad B_j=-\frac{c_j}{2}\frac{m}{A_j}.\quad (j=1,2,3) \tag{4.5}$$

The first of the above equations is identical with the conservation equation of the impulse of the dispersion phase (3.2) regarded as a continuous medium in the equilibrium state.

Computing formally $\langle w_i'^2\rangle$ of (4.4) and (4.5), the following relations are obtained:

$$\frac{A_i}{m}=-c_i\langle w_i'^2\rangle_0,\quad \frac{\mathrm{tr}\,\mathbf{A}}{m}=c\theta_0,\quad c=-\sum c_jM_j,\quad B_j=\frac{1}{2M_j\theta_0}$$
$$M_1=\frac{1}{1+2N_w},\quad M_2=\frac{N_w}{1+2N_w}=M_3 \tag{4.6}$$

As it was said before, the quantities $\langle w_i'^2\rangle_0$, θ_0 and the coefficients c_j, M_j may be regarded as known functions of the dynamic variables. Therefore the relations (4.6) determine finally the kinetic equation (4.1) and the equilibrium-distribution function.

Using familiar approach, the following conservation equation for the quantity θ is obtained from (4.1):

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)(\rho\theta)+\rho\theta\frac{\partial \mathbf{w}}{\partial \mathbf{r}}\frac{\partial \mathbf{q}_\theta}{\partial \mathbf{r}}-2c\rho\theta+\frac{2}{d_2}\mathbf{P}^{(p)}:\left(\frac{\partial}{\partial \mathbf{r}}*\mathbf{w}\right)=2\rho\frac{\mathrm{tr}\,\mathbf{A}}{m},$$
$$\mathbf{q}_\theta=\sigma_0\int w'^2\mathbf{w}'f d\mathbf{w}',\quad \sigma_0=\frac{4}{3}\pi a^3 \tag{4.7}$$

If the equation is multiplied by $\frac{1}{2}m$, then it obviously becomes the transfer equation of the pseudoturbulent energy of particles, which is similar to the heat-conduction equation. Then the quantity $\frac{1}{2}m\mathbf{q}_\theta$ represents the pseudoturbulent flow of the energy.

Using (4.6) we write (4.7) in a different form, namely,

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)(\rho\theta)+\rho\theta\frac{\partial \mathbf{w}}{\partial \mathbf{r}}+\frac{\partial \mathbf{q}_\theta}{\partial \mathbf{r}}+\frac{2}{d_2}\mathbf{P}^{(p)}:\left(\frac{\partial}{\partial \mathbf{r}}*\mathbf{w}\right)=2c\rho\left(\theta_0-\theta\right) \tag{4.8}$$

Equations (4.8) and (3.2) represent a complete system of equations which determine the average motion of suspension in the continuous approximation. All the characteristics of the pseudoturbulence which appear in these equations are determined in terms of θ and of the dynamic variables in accordance with (3.1) and the results of Sec. 2. The quantity $\mathbf{q}_\theta$ is the only exception and can be computed only if the distribution function under nonequilibrium conditions is known.

The solution f of Eq. (4.1) and the corresponding expression for $\mathbf{q}_\theta$ can be sought formally in the form of a power series:

$$f=\sum f^{(r)}\varepsilon^r,\qquad \mathbf{q}_\theta=\sum \mathbf{q}_\theta^{(r)}\varepsilon^r \tag{4.9}$$

in which ε denotes a small quantity of the order of the ratio of the pseudoturbulence scale to the scale of the corresponding average motion.

As in (4.9), the zeroth term in the expansion of f in powers of ε in the Euler approximation considered by us here is simply identical with the function $f^{(0)}$ in (4.4); the corresponding term $\mathbf{q}_\theta{}^{(0)}$ in the power series for $\mathbf{q}_\theta$ vanishes identically. The next term in the expansion of $\mathbf{q}_\theta$ is obviously of the order of ε and need not therefore be taken into account in the Euler approximation. Thus in Eq. (4.8) one should adopt $\mathbf{q}_\theta = 0$.

In all subsequent approximations the individual coefficients $f^{(r)}$ in (4.9) also depend on ε; this is due to the presence of derivatives of dynamic variables in the complete stochastic equations of [1], resulting in the dependence on ρ of all characteristics of the equilibrium pseudoturbulence and all components of the tensor $\mathbf{A}$ in the kinetic equation (4.1). It can be seen that for such approximations further terms of the series (4.9) must be calculated up to the index r determined by the order of approximation. By analogy with the kinetic theory of gases and hydrodynamic approximations of the first or second orders in ε it seem appropriate to call them Navier-Stokes and Barnett hydrodynamic approximations for suspensions.

It should be mentioned that the solutions θ either of Eqs. (4.7) or (4.8) are, as can easily be seen, stable only if $c > 0$.

Thus, the following nonvanishing characteristics of the equilibrium pseudoturbulence appear in Eqs. (4.8) and (3.2), which determine the motion of the suspension in Euler approximation:

$$P_{ij}^{(p)} \sim \langle w_i' w_j' \rangle \rho,\qquad \langle v_i' v_j' \rangle (1-\rho),\qquad q_1\lambda_K,\ c \tag{4.10}$$

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